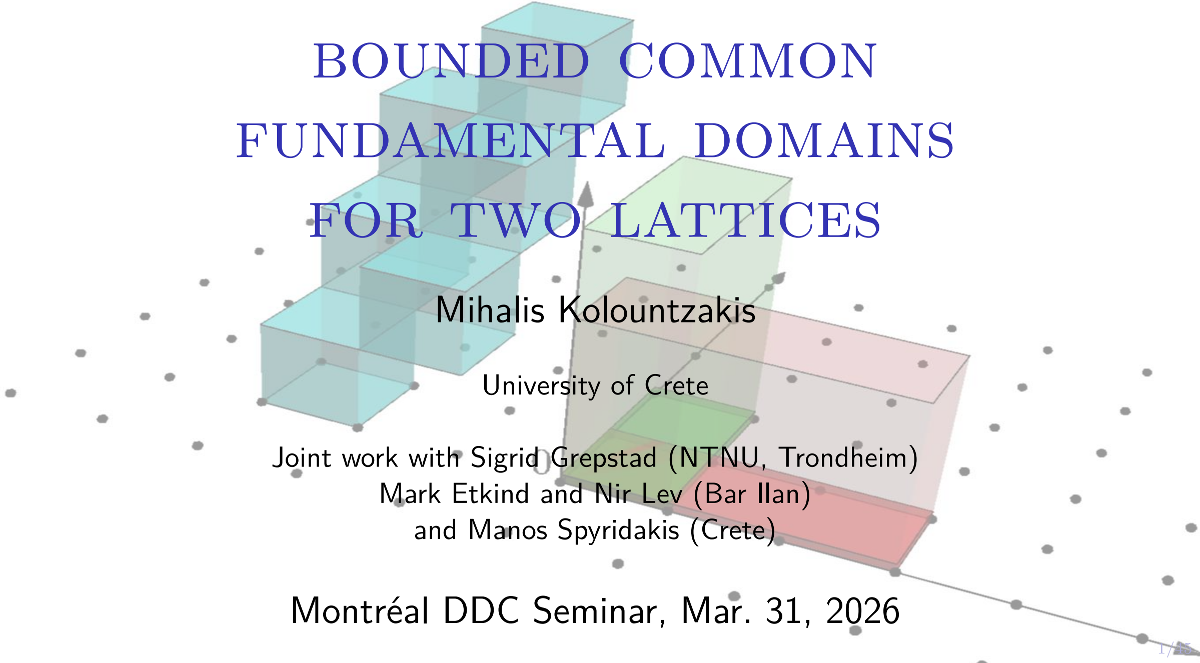

BOUNDED COMMON FUNDAMENTAL DOMAINS FOR TWO LATTICES

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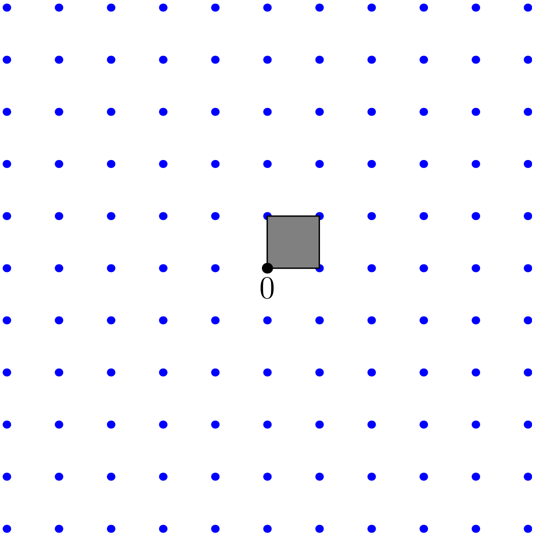
Joint work with Sigrid Grepstad (NTNU, Trondheim)
Mark Etkind and Nir Lev (Bar Ilan)
and Manos Spyridakis (Crete)

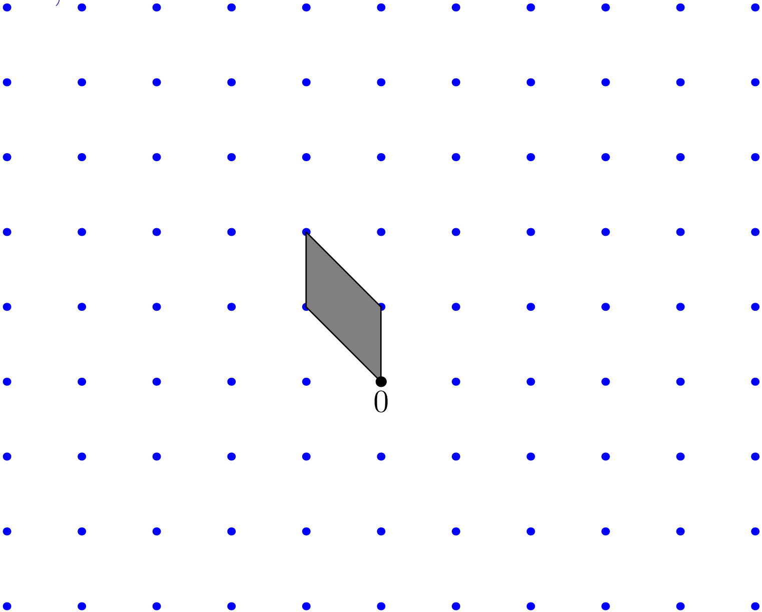
Montréal DDC Seminar, Mar. 31, 2026

A lattice and a fundamental domain

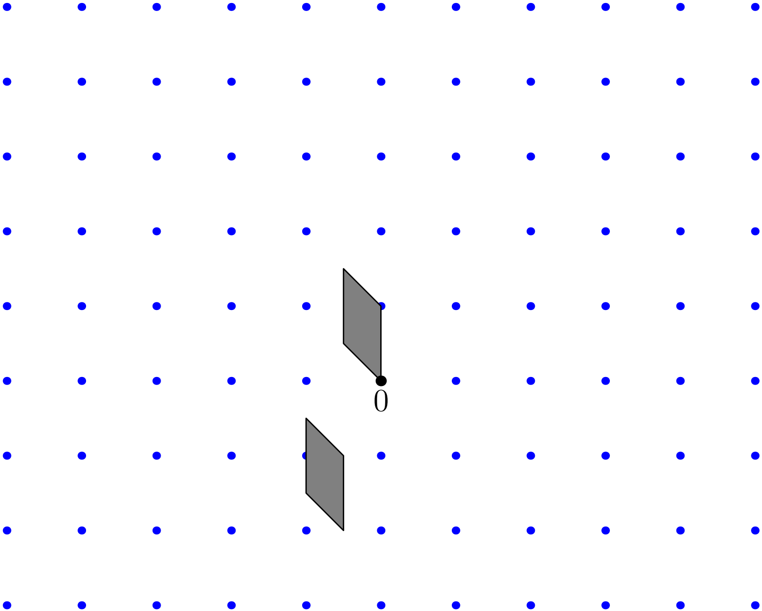


$Q = [0,1)^d$ or $[0,1]^d$ (ignoring measure 0)

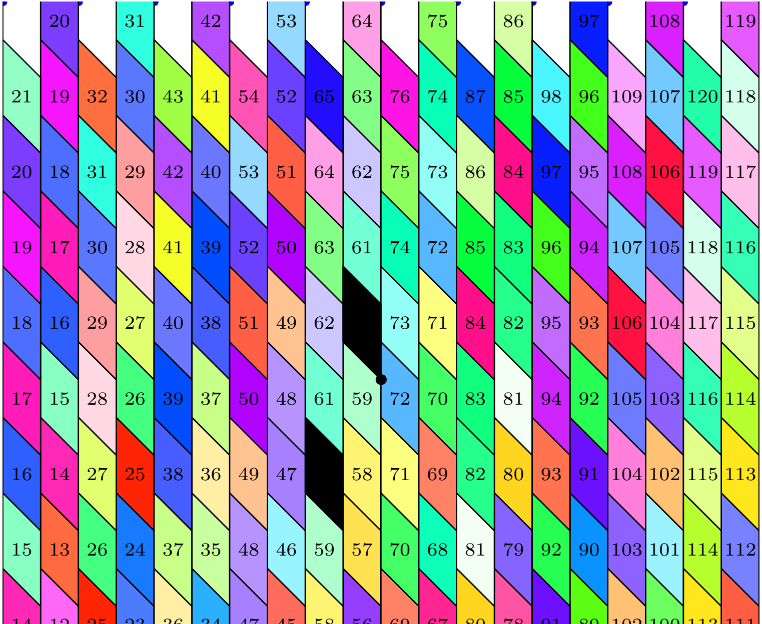
Same lattice, another fundamental domain



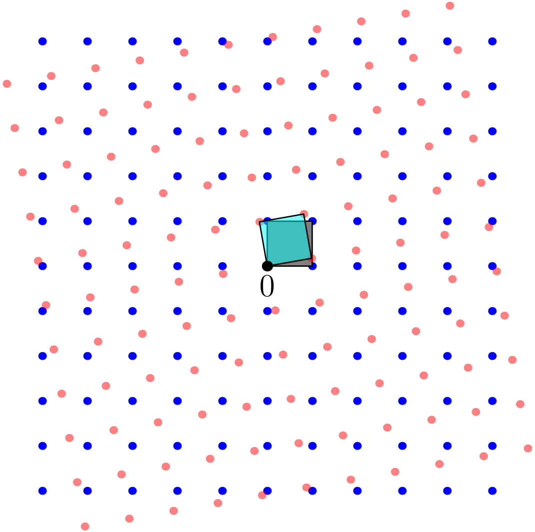
Yet another fundamental domain



And its tiling



Two lattices, their fundamental domains



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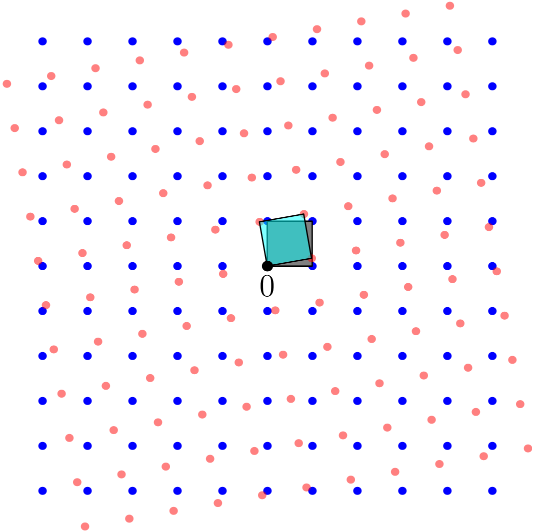
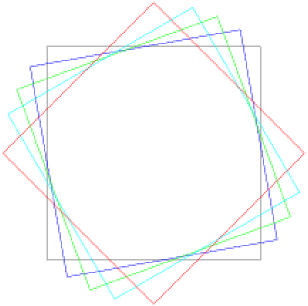


Main question of interest: can we have a common fundamental domain?

An extreme case: the Steinhaus tiling problem

- Can we have $A + \rho\mathbb{Z}^2 = \mathbb{R}^2$ be a tiling for all rotations ρ?



Can a domain behave simultaneously like all those squares?

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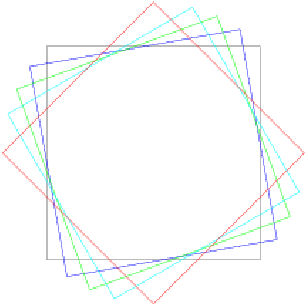

Can a domain behave simultaneously like all those squares?

- Problem still open in dimension 2 for measurable sets (ignoring measure 0)
  Without measurability: answer is yes (Jackson and Mauldin, 2000).

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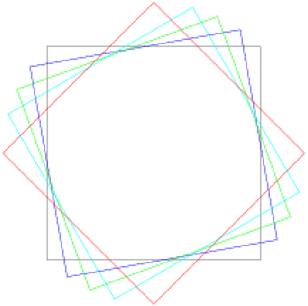

Can a domain behave simultaneously like all those squares?

- Problem still open in dimension 2 for measurable sets (ignoring measure 0)
  Without measurability: answer is yes (Jackson and Mauldin, 2000).
- In dimension 3 answer is no for measurable sets
  (K. & Wolff, 1997, K. & Papadimitrakis, 2002).
  Unknown without measurability.

Lattice Steinhaus for finitely many lattices (K. 1997)

- Given lattices

$$\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{R}^d$$

all of volume 1.
Find measurable A which tiles a.e.
with <u>all</u> Λ_j.

Lattice Steinhaus for finitely many lattices (K. 1997)

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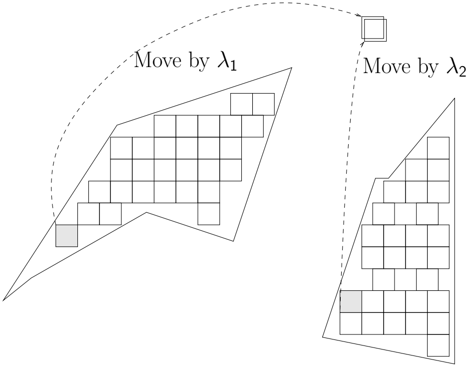
  Find measurable A which tiles a.e.
  with <u>all</u> Λ_j.

- Generically yes!
  If the sum

$$\Lambda_1^* + \cdots + \Lambda_n^*$$

  is direct then Kronecker-type density theorems allow us to rearrange a fundamental domain of one lattice to accommodate the others.



An application in Gabor analysis

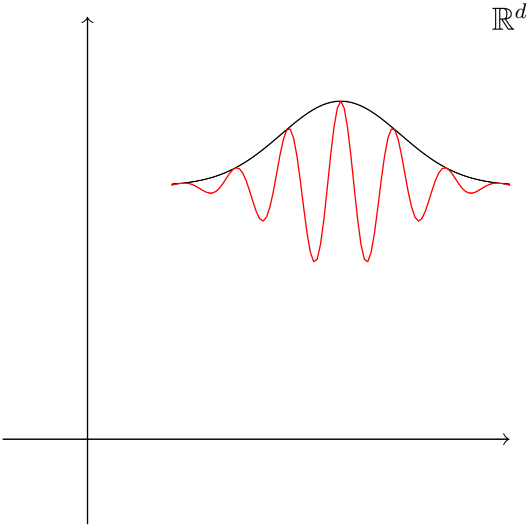
- **Problem:** If K, L are two lattices in $\mathbb{R}^d$ with

$$\operatorname{vol} K \cdot \operatorname{vol} L = 1,$$

can we find $g \in L^2(\mathbb{R}^d)$, such that the (K, L) time-frequency translates

$$g(x-k)e^{2\pi i \ell \cdot x}, \quad (k \in K, \ell \in L)$$

form an orthogonal basis of $L^2(\mathbb{R}^d)$?



A time-frequency translate of g.

An application in Gabor analysis (continued)

- Han and Wang (2000):
  Since $\mathrm{vol}\,(L^*) = \mathrm{vol}\,(K)$ let $g = \mathbf{1}_E$
  where
  E is a **common tile** for K, L^*.
- $E(L) = \left\{e^{2\pi i \ell \cdot x} : \ell \in L\right\}$ forms an orthogonal basis for any FD of L^*, so of $L^2(E+x)$ (for any x).

An application in Gabor analysis (continued)

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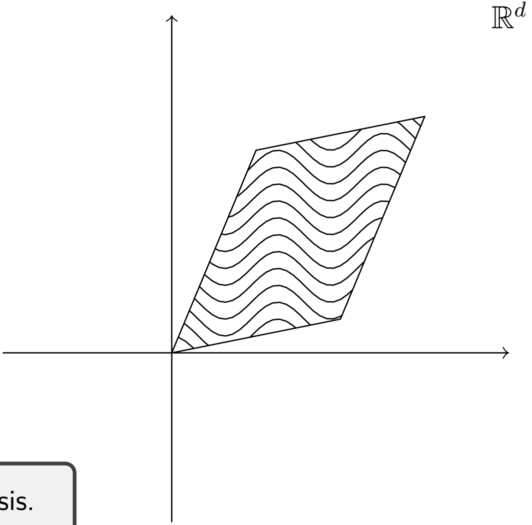
- K-tiling $\Longrightarrow$

  Partition space in K-translates of E

- L^*-tiling $\Longrightarrow$

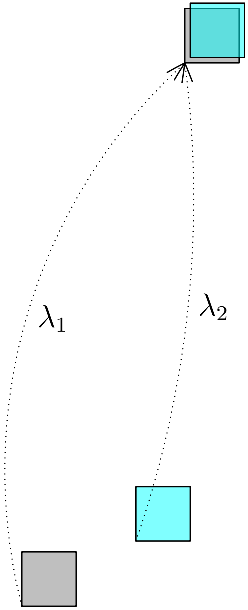
  On each copy $E(L)$ is an orthogonal basis.



A time-frequency translate of $\mathbf{1}_E$.

Unbounded common fundamental domains

- Using the density of $\Lambda_1 + \Lambda_2$ leads to unbounded common domains.
- *Problem with density approach:* Pieces of the two fundamental domains may have to be moved very far in order to achieve high overlap.
- It has been an open problem for more than 25 years if a **bounded** common fundamental domain exists.



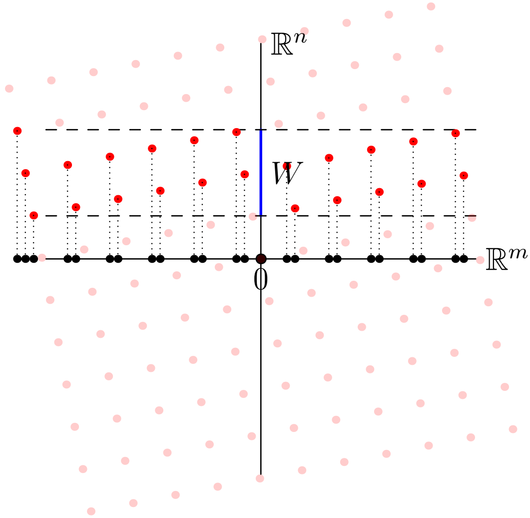
Bounded common FDs always exist for two lattices

This is our main result.

Theorem 1 (S. Grepstad and K., 2025)

If L, M are two lattices in $\mathbb{R}^d$ of the same volume then they have a common, measurable and bounded fundamental domain.

The main new tool are some properties of cut-and-project sets.



A cut-and-project set on $\mathbb{R}^m$.

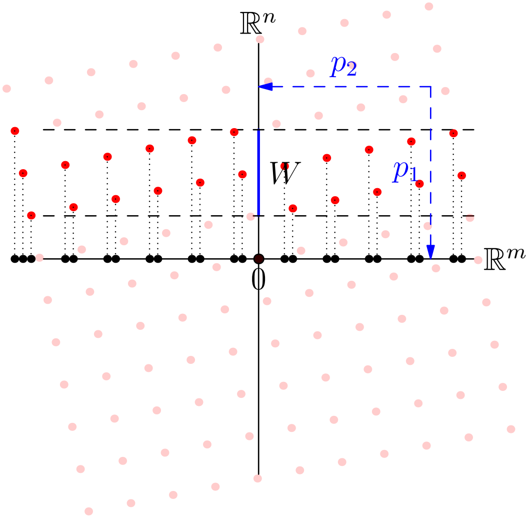
Cut-and-project (model) sets

- Lattice $\Gamma \subseteq \mathbb{R}^m \times \mathbb{R}^n$
  Window $W \subseteq \mathbb{R}^n$.
- Projection p_1 injective on Γ
- $p_2(\Gamma)$ dense in $\mathbb{R}^n$
- *Model set*:

$$\Lambda(\Gamma, W) = \{p_1(\gamma) : \gamma \in \Gamma,\ p_2(\gamma) \in W\}$$

It is called *regular* if $|\partial W| = 0$. Then

$$\operatorname{dens} \Lambda(\Gamma, W) = |W| \cdot \operatorname{dens} \Gamma.$$



Lattice $\Gamma \subseteq \mathbb{R}^m \times \mathbb{R}^n$, window $W \subseteq \mathbb{R}^n$.
Model set $\Lambda(\Gamma, W) \subseteq \mathbb{R}^m$.

Bounded distance equivalence (BDE)

- $A, B \subseteq \mathbb{R}^d$ are *bounded distance equivalent* (BDE) if there is

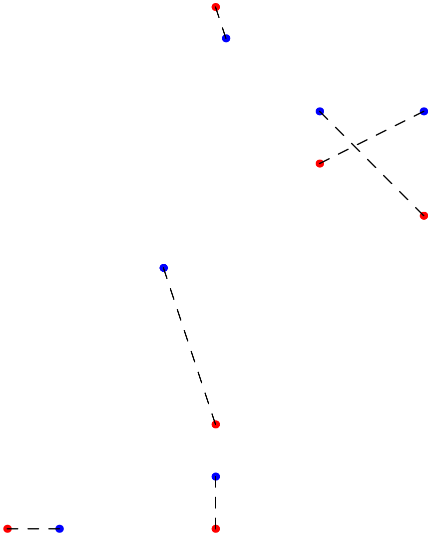
  $$\text{bijection } \phi : A \to B$$

  with

  $$\sup_{a \in A} \{|a - \phi(a)|\} < \infty.$$

- A and B must have the same density.
- *A consequence of Hall's marriage theorem:*

  Any two *lattices* of the same density are BDE.



Non-measurable bounded common FDs when $L \cap M = \{0\}$

- Suppose $L, M \subseteq \mathbb{R}^d$ lattices with $L \cap M = \{0\}$.
- A common set of coset representatives of L, M in $L + M$ is

$$S = \{\ell_i - m_i : i \in \mathbb{N}\}$$

where ℓ_i, m_i any enumeration of L, M.

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- Choose the enumeration so that $\phi(\ell_i) = m_i$. Then S is bounded.

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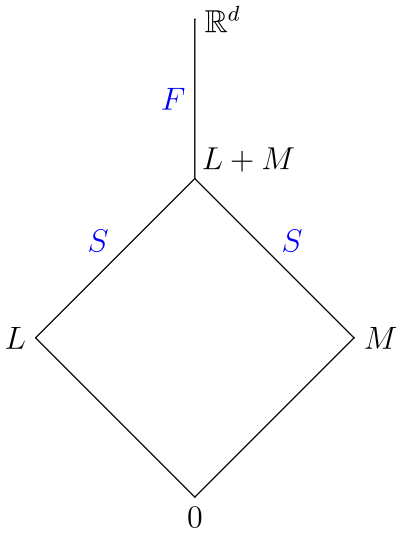
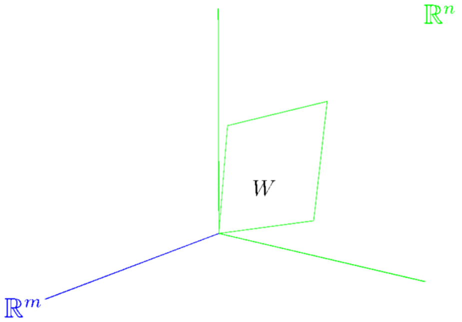
- $\mathrm{dens}\, L = \mathrm{dens}\, M$ implies L, M are BDE.
- Choose the enumeration so that $\phi(\ell_i) = m_i$. Then S is bounded.
- Can choose bounded F a FD of $L + M$ in $\mathbb{R}^d$. It need not be measurable (e.g. when $L + M$ is dense).
- Then $S + F$ is a bounded, common FD of L, M in $\mathbb{R}^d$.



Model sets from parallelepipeds and BDE

Theorem 2 (Duneau and Oguey, 1991)

If $W \subseteq \mathbb{R}^n$ is a parallelepiped spanned by vectors in $p_2(\Gamma)$ then the model set $\Lambda(\Gamma, W)$ is BDE to some lattice in $\mathbb{R}^m$.



Equidecomposable sets and BDE

- $A, B \subseteq \mathbb{R}^d$ are *equidecomposable* if can be partitioned

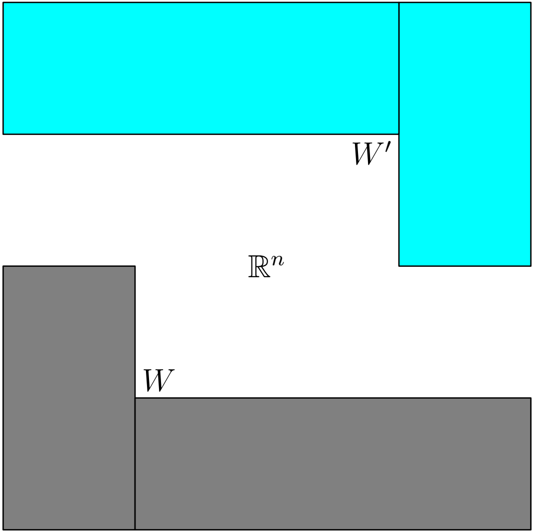
$$A = \bigcup_{i=1}^{N} A_i, \quad B = \bigcup_{i=1}^{N} B_i$$

such that $A_i = B_i + \tau_i$.
If all τ_i are in a group $G \subseteq \mathbb{R}^d$ then they are *G-equidecomposable*.

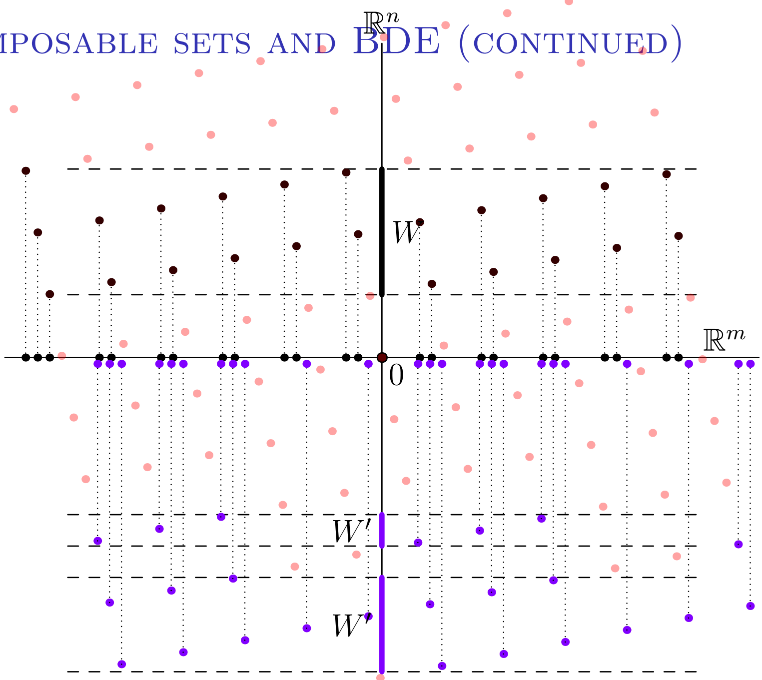
Theorem 3 (Frettlöh and Garber, 2018)

If the windows $W, W' \subseteq \mathbb{R}^n$ are $p_2(\Gamma)$-equidecomposable then

$$\Lambda(\Gamma, W), \Lambda(\Gamma, W') \text{ are BDE.}$$



Equidecomposable sets and BDE (continued)



From BDE to equidecomposable windows

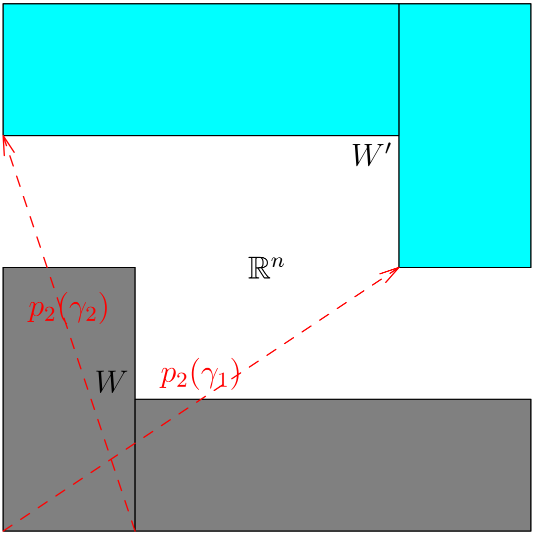
- Suppose $W, W' \subseteq \mathbb{R}^n$ measurable, bounded, and $|\partial W| = |\partial W'| = 0$.

Theorem 4 (Grepstad, 2024)

If the model sets

$$\Lambda(\Gamma, W), \Lambda(\Gamma, W') \text{ are BDE}$$

then W, W' are $p_2(\Gamma)$-equidecomposable up to measure 0.



Bounded common FD when $L + M$ is dense

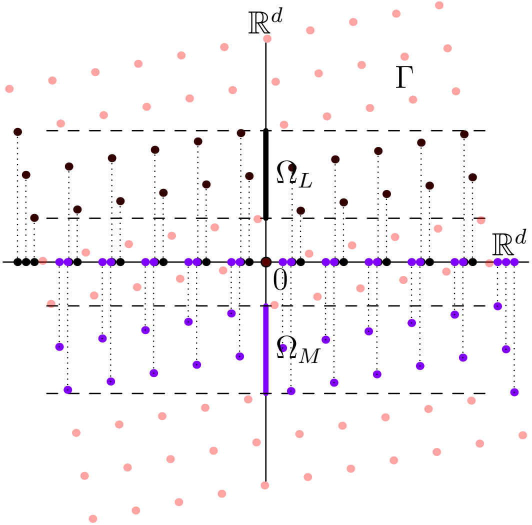
- Let $L = L\mathbb{Z}^d$, $M = M\mathbb{Z}^d$ have $\overline{L + M} = \mathbb{R}^d$.
- $\Omega_L, \Omega_M \subseteq \mathbb{R}^d$ fundamental parellelepipeds of same volume.

Bounded common FD when $L + M$ is dense

- Let $\mathcal{L} = L\mathbb{Z}^d$, $\mathcal{M} = M\mathbb{Z}^d$ have $\overline{\mathcal{L} + \mathcal{M}} = \mathbb{R}^d$.
- $\Omega_L, \Omega_M \subseteq \mathbb{R}^d$ fundamental parellelepipeds of same volume.
- Lattice $\Gamma = \Gamma\mathbb{Z}^{2d}$ defined by

$$\Gamma = \begin{bmatrix} \multicolumn{2}{c}{K} \\ \hline L & M \end{bmatrix}$$

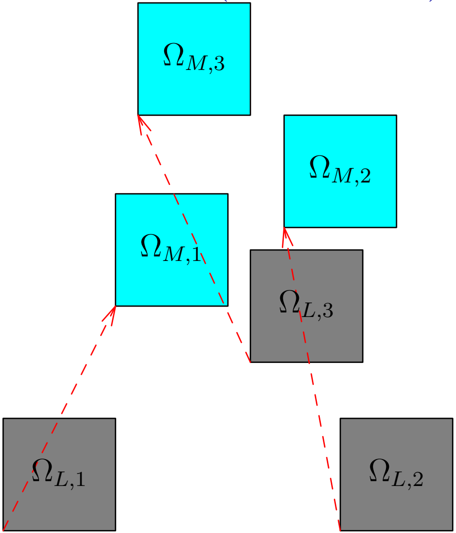
where $K \in \mathbb{R}^{d\times 2d}$ is injective on $\mathbb{Z}^{2d}$ (e.g. random) and $\det\Gamma \neq 0$.



$p_1|_\Gamma$ is injective.
$p_2(\Gamma) = \mathcal{L} + \mathcal{M}$ is dense in $\mathbb{R}^d$.

Bounded common FD when $L + M$ is dense (continued)

- Parallelepipeds Ω_L, Ω_M are spanned by vectors in $p_2(\Gamma) = L + M$.
- **Theorem 2** $\implies \Lambda(\Gamma, \Omega_L), \Lambda(\Gamma, \Omega_M)$ are each BDE to some lattices L', M'.
- Lattice densities are $|\Omega_L| = |\Omega_M|$.



The translation vectors $\Omega_{L,i} \to \Omega_{M,i}$ are in $L + M$.

Bounded common FD when $L+M$ is dense (continued)

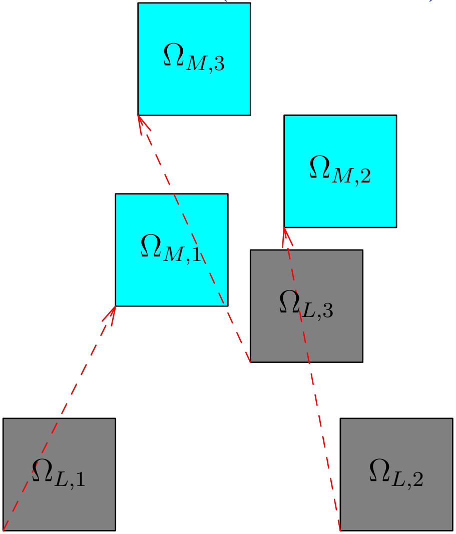
- Parallelepipeds Ω_L, Ω_M are spanned by vectors in $p_2(\Gamma) = L+M$.
- **Theorem 2** $\implies \Lambda(\Gamma,\Omega_L), \Lambda(\Gamma,\Omega_M)$ are each BDE to some lattices L', M'.
- Lattice densities are $|\Omega_L| = |\Omega_M|$.
- Lattices of equal density are BDE, so

$$\Lambda(\Gamma,\Omega_L) \overset{\text{BDE}}{\sim} L' \overset{\text{BDE}}{\sim} M' \overset{\text{BDE}}{\sim} \Lambda(\Gamma,\Omega_M).$$

- **Theorem 4** $\implies \Omega_L, \Omega_M$ are $p_2(\Gamma)$-equidecomposable a.e.

$$\Omega_L = \bigcup_{i=1}^{N} \Omega_{L,i}, \quad \Omega_M = \bigcup_{i=1}^{N} \Omega_{M,i},$$

$$\Omega_{L,i}. + \tau_i = \Omega_{M,i}, \quad \tau_i \in L+M.$$



The translation vectors $\Omega_{L,i} \to \Omega_{M,i}$ are in $L+M$.

Bounded common FD when $L+M$ is dense (conclusion)

- $\Omega_{L,i}. + \tau_i = \Omega_{M,i}, \quad \tau_i \in L+M$, so

$$\Omega_{L_i} + \ell_i = \Omega_{M,i} - m_i$$

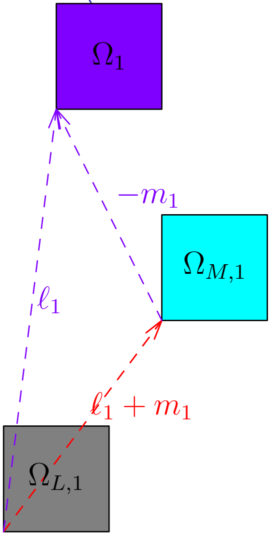
for some $\ell_i \in L$, $m_i \in M$.

- Translation by ℓ_i does not change class mod L, and translation by $-m_i$ does not change class mod M.

-

$$\Omega = \bigcup_{i=1}^{N} \Omega_{L,i} + \ell_i = \bigcup_{i=1}^{N} \Omega_{M,i} - m_i$$

is a bounded, common FD for L, M.



How the parts $\Omega_{L,1}$ and $\Omega_{M,1}$ merge into a single Ω_1.

Bounded common FD when $L + M$ is discrete

Lemma 5

If G_1, G_2 subgroups of G of same index k there exist $g_1, \ldots, g_k \in G$ which are coset representatives for both G_1 and G_2.

Bounded common FD when $L + M$ is discrete

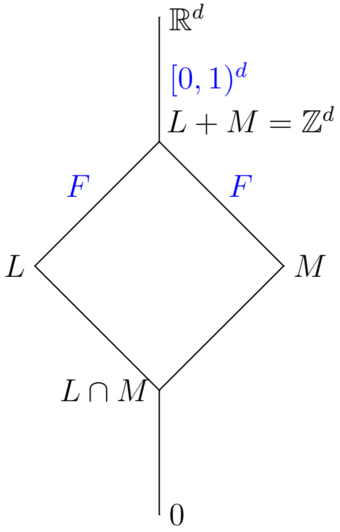
Lemma 5

If G_1, G_2 subgroups of G of same index k there exist $g_1, \ldots, g_k \in G$ which are coset representatives for both G_1 and G_2.

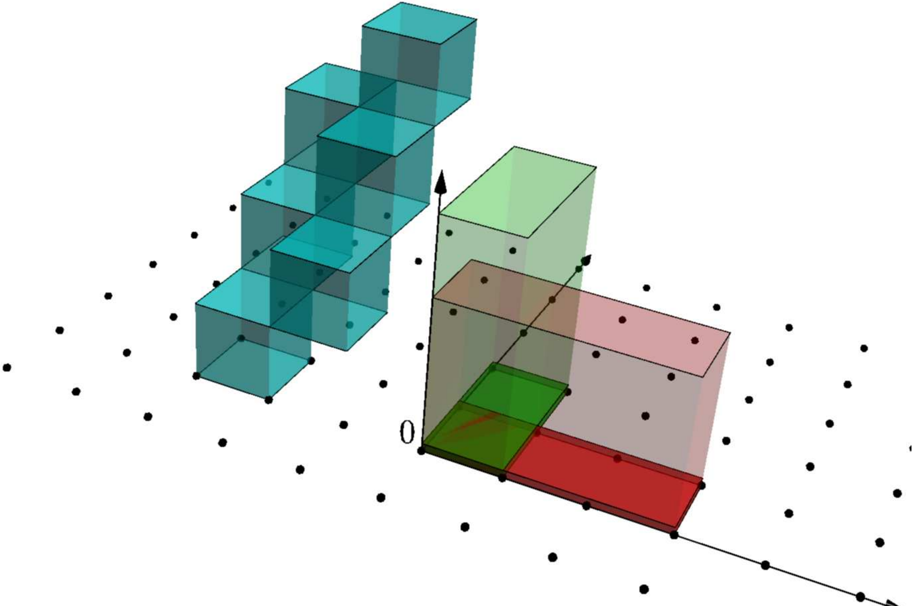
- May assume $L + M = \mathbb{Z}^d$.
- The Lemma gives finite $F \subseteq \mathbb{Z}^d$ s.t. $L + F = \mathbb{Z}^d$ and $M + F = \mathbb{Z}^d$ are both tilings. Then

$$L + F + [0,1]^d = \mathbb{R}^d \text{ and } M + F + [0,1]^d = \mathbb{R}^d$$

are both tilings, so $F + [0,1]^d$ is a bounded, common FD for L, M.



Bounded common FD: a case with $L + M$ discrete



Bounded common FD: intermediate case

- $\overline{L+M} \subseteq \mathbb{R}^d$ is always a linear image of

$$\mathbb{Z}^m \times \mathbb{R}^n, \text{ with } m+n=d.$$

- Cases remaining: $m > 0, n < d$.
- Define the groups

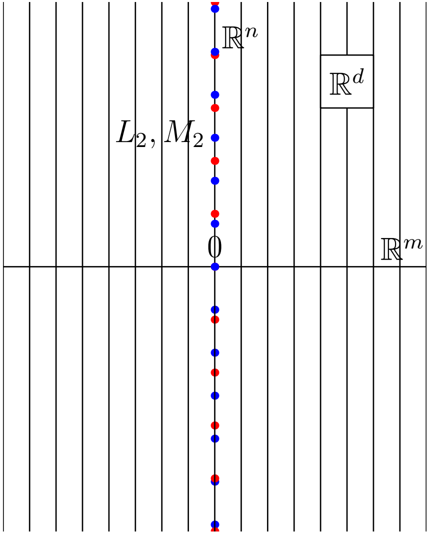
$$L_2 = (\{0\}^m \times \mathbb{R}^n) \cap L \text{ and } M_2 = (\{0\}^m \times \mathbb{R}^n) \cap M.$$

These are lattices of rank n.

- Complete L_2, M_2 as follows:

$$L = L_1 \oplus L_2, \quad M = M_1 \oplus M_2,$$

where $L_1, M_1 \subseteq \mathbb{Z}^m \times \mathbb{R}^n$, have rank m.



The group $\overline{L+M}$. Lattices $L_2, M_2 \subseteq \mathbb{R}^n$ shown in red, blue.

Bounded common FD: intermediate case (continued)

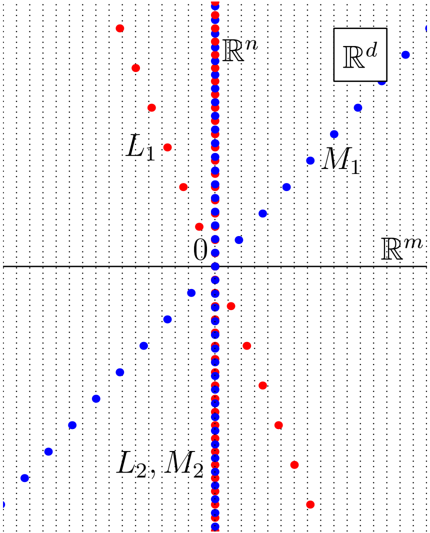
- We have

$$[\mathbb{Z}^m \times \mathbb{R}^n : L_1 \oplus \{0\}^m \times \mathbb{R}^n] = \det L_1$$
$$[\mathbb{Z}^m \times \mathbb{R}^n : M_1 \oplus \{0\}^m \times \mathbb{R}^n] = \det M_1.$$

  and
  $L_1 + M_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n$.

- Can pick the representing matrices
  $$L = \begin{pmatrix} L_1 & 0 \\ * & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_1 & 0 \\ * & M_2 \end{pmatrix}.$$
  and $\det L_1, \det M_1 \in \mathbb{Z}$ since $L_1, M_1 \subseteq \mathbb{Z}^{m\times m}$.

- Since
  $\det L = \det L_1 \cdot \det L_2, \ \det M = \det M_1 \cdot \det M_2.$
  we have
  $$\frac{\det L_2}{\det M_2} = \frac{\det M_1}{\det L_1} \in \mathbb{Q}.$$



Bounded common FD: intermediate case (continued)

- Pick superlattices $L_2', M_2' \subseteq \{0\}^m \times \mathbb{R}^n$ s.t.

  $$[L_2' : L_2] = \det M_1 \quad \text{and} \quad [M_2' : M_2] = \det L_1.$$

  Then $\det L_2' = \det M_2'$.

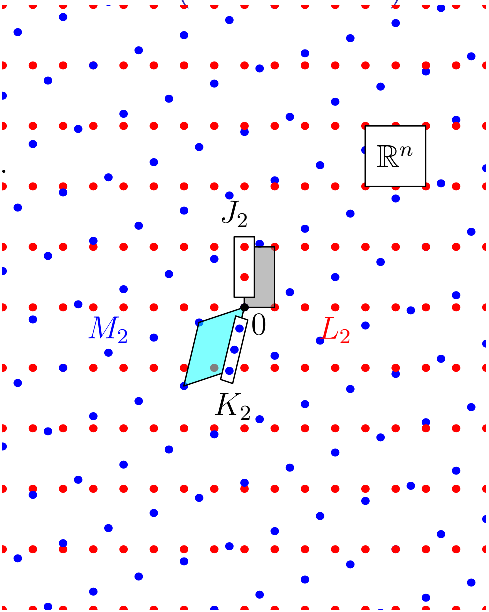
- Since $\overline{L_2' + M_2'} = \mathbb{R}^n$ there is bounded $E' \subseteq \mathbb{R}^n$:

  $$E' + L_2' = E' + M_2' = \mathbb{R}^n \quad \text{are tilings.}$$

- Pick finite $J_2 \subseteq L_2'$, $K_2 \subseteq M_2'$ s.t.

  $$L_2' = L_2 + J_2, \ \ M_2' = M_2 + K_2 \quad \text{are tilings.}$$

  Then $|J_2| = \det M_1$, $|K_2| = \det L_1$.



Bounded common FD: intermediate case (continued)

- Since (not a tiling below)

$$L_1 + M_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n$$

  we can pick finite sets $J_1 \subseteq L_1$, $K_1 \subseteq M_1$

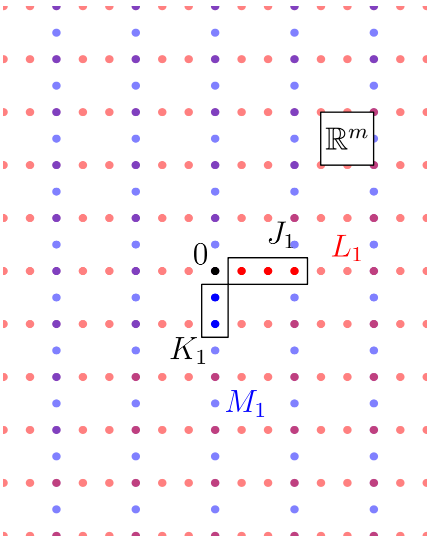
$$|J_1| = \det M_1, \quad |K_1| = \det L_1$$

  s.t. we have the tilings

$$K_1 + L_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n$$
$$J_1 + M_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n.$$

- $|J_1| = |J_2|$ and $|K_1| = |K_2|$ so

$$\exists \text{ bijections } \phi : K_1 \to K_2, \quad \psi : J_1 \to J_2.$$



Bounded common FD: intermediate case (continued)

- Remember

$$E' + L_2' = E' + M_2' = \mathbb{R}^n \quad \text{are tilings.}$$

- Define the bounded set $E \subseteq \mathbb{Z}^m \times \mathbb{R}^n$

$$E = \{x + y + \phi(x) + \psi(y) : x \in K_1,\ y \in J_1\} \ + \ E'.$$

  Claim:

$$L + E = M + E = \mathbb{Z}^m \times \mathbb{R}^n \text{ are tilings.}$$

- Then

$$\Omega = E + [0,1]^m \times \{0\}^n$$

  is our bounded, common tile for L, M.

- By periodicity of the arrangement and matching of volumes it suffices to prove

$$L + E = \mathbb{Z}^m \times \mathbb{R}^n \text{ and } M + E = \mathbb{Z}^m \times \mathbb{R}^n \text{ are packings.}$$

Bounded common FD: intermediate case (continued)

- Suppose not: $\ell = \ell_1 + \ell_2$ and $\widetilde{\ell} = \widetilde{\ell}_1 + \widetilde{\ell}_2 \in L = L_1 \oplus L_2$ and $\ell + E$ and $\widetilde{\ell} + E$ overlap.

$$\exists\ x, \widetilde{x} \in K_1, \quad y, \widetilde{y} \in J_1 \text{ such that}$$

$$\ell_1 + \ell_2 + x + y + \phi(x) + \psi(y) + E', \ \widetilde{\ell}_1 + \widetilde{\ell}_2 + \widetilde{x} + \widetilde{y} + \phi(\widetilde{x}) + \psi(\widetilde{y}) + E' \text{ overlap.}$$

- Rewrite overlapping copies as

$$\underbrace{\ell_1 + y}_{\in L_1} + x + \underbrace{\ell_2 + \phi(x) + \psi(y) + E'}_{\subseteq \{0\}^m \times \mathbb{R}^n}$$

and

$$\underbrace{\widetilde{\ell}_1 + \widetilde{y}}_{\in L_1} + \widetilde{x} + \underbrace{\widetilde{\ell}_2 + \phi(\widetilde{x}) + \psi(\widetilde{y}) + E'}_{\subseteq \{0\}^m \times \mathbb{R}^n}.$$

- Remember the tiling $K_1 + L_1 + \{0\}^m \times \mathbb{R}^n = \mathbb{Z}^m \times \mathbb{R}^n$. Since $x, \widetilde{x} \in K_1$ we get

$$\ell_1 + y = \widetilde{\ell}_1 + \widetilde{y} \quad \text{and} \quad x = \widetilde{x},$$

so $\phi(x) = \phi(\widetilde{x})$.

Bounded common FD: intermediate case (continued)

- Thus the translates

$$\ell_2 + \psi(y) + E' \quad \text{and} \quad \widetilde{\ell}_2 + \psi(\widetilde{y}) + E'$$

overlap on positive measure.

- But $\ell_2 + \psi(y), \widetilde{\ell}_2 + \psi(\widetilde{y}) \in L_2'$ and

$$\mathbb{R}^n = L_2' + E' = L_2 + J_2 + E' \text{ are tilings,}$$

so we get $\ell_2 = \widetilde{\ell}_2$ and $\psi(y) = \psi(\widetilde{y})$.

- Then $y = \widetilde{y}$ since ψ is a bijection.
- Finally from $\ell_1 + y = \widetilde{\ell}_1 + \widetilde{y}$ we obtain

$$\ell_1 = \widetilde{\ell}_1.$$

Bounded common FD: intermediate case (conclusion)

- Packing implies tiling since

$$\begin{aligned} |E| &= \left|E'\right| \cdot |K_1| \cdot |J_1| \\ &= \det L_2' \cdot \det L_1 \cdot \det M_1 \\ &= \det L_1 \cdot \det L_2' \cdot |J_2| \\ &= \det L_1 \cdot \det L_2 \\ &= \det L. \end{aligned}$$

- Same result for $M + E$ by symmetry.
- End of proof.

THE RESULT THAT ENABLED THIS CONSTRUCTION

THEOREM 4 (GREPSTAD, 2024)

If the model sets

$$\Lambda(\Gamma, W), \Lambda(\Gamma, W') \text{ are BDE}$$

then W, W' are $p_2(\Gamma)$-equidecomposable up to measure 0.

And a mid-summer (2025) emergency!

Theorem 4 (Grepstad, 2024)

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?

Error in proof pointed out by Nir Lev.

Our savior

J. Eur. Math. Soc. 24, 2751–2773 (2022) DOI 10.4171/JEMS/1164

Tomasz Cieśla · Marcin Sabok

Measurable Hall's theorem for actions of abelian groups

Received May 28, 2019; revised June 29, 2021

Abstract. We prove a measurable version of the Hall marriage theorem for actions of finitely generated abelian groups. In particular, it implies that for free measure-preserving actions of such groups and measurable sets which are suitably equidistributed with respect to the action, if they are are equidecomposable, then they are equidecomposable using measurable pieces. The latter generalizes a recent result of Grabowski, Máthé and Pikhurko on the measurable circle squaring and confirms a special case of a conjecture of Gardner.

Keywords. Circle squaring, Hall matching theorem, Mokobodzki medial means

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Immeasurable gratitude!

Measurable Hall's theorem

Theorem 2. *Let Γ be a finitely generated abelian group and let $\Gamma \curvearrowright (X, \mu)$ be a free pmp action. Suppose $A, B \subseteq X$ are two measurable Γ-uniform sets. The following are equivalent:*

(1) *The pair A, B satisfies the Hall condition with respect to Γ μ-a.e.*

(2) *A and B are Γ-equidecomposable μ-a.e. using μ-measurable sets.*

(3) *A and B are Γ-equidecomposable μ-a.e.*

- We will show (3).

The corrected theorem

Theorem 6 (M. Etkind, S. Grepstad, M.K. and N. Lev (2025))

Assume

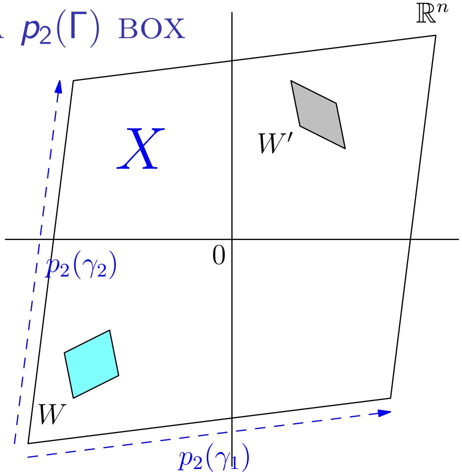
- $W, W' \subseteq \mathbb{R}^n$ *are bounded measurable sets of positive measure.*
- $|\partial W| = |\partial W'| = 0$
- *The model sets* $\Lambda(\Gamma, W)$ *and* $\Lambda(\Gamma, W')$ *are bounded distance equivalent.*

Then W, W' *are* $p_2(\Gamma)$*-equidecomposable up to measure zero with* ***measurable*** *pieces.*

Plan of proof

- First prove $p_2(\Gamma)$-equidecomposability with *any* pieces.
- Then use measurable Hall to get measurable pieces.

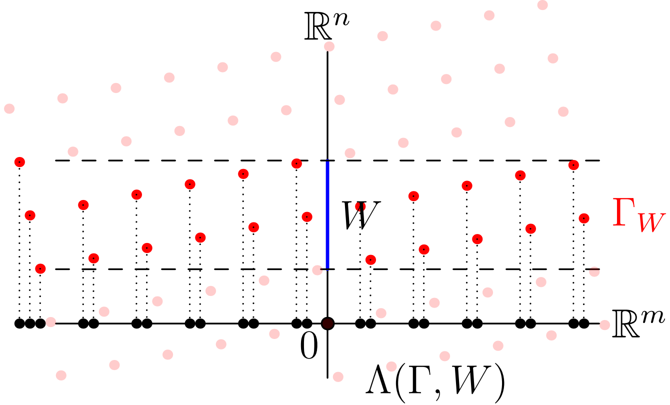
Enclose W, W' in a $p_2(\Gamma)$ box



- Define a sparse lattice $H \subseteq p_2(\Gamma)$ and the quotient space $X = \mathbb{R}^n / H$
- $G = p_2(\Gamma)/H$ acts on X by translation.
- W, W' are viewed as subsets of X.
- We show W and W' are $p_2(F)$-equidecomposable for some *finite* $F \subseteq \Gamma$.

Set Γ_W which p_1-projects to $\Lambda(\Gamma, W)$

- Define $\Gamma_W = \{\gamma \in \Gamma : p_2(\gamma) \in W\}, \quad \Gamma_{W'} = \{\gamma \in \Gamma : p_2(\gamma) \in W'\}$



- Easy: $\Lambda(\Gamma, W) \overset{\text{BDE}}{\sim} \Lambda(\Gamma, W') \implies \Gamma_W \overset{\text{BDE}}{\sim} \Gamma_{W'}$

Remove the kernel of p_2 on Γ

- If $N = \ker p_2 \cap \Gamma = (\mathbb{R}^m \times \{0\}^n) \cap \Gamma$
  then we have the lattice decomposition $\Gamma = L \oplus N$ and

$$\Gamma_W = L_W \oplus N, \quad \Gamma_{W'} = L_{W'} \oplus N$$

  where $L_W = \{\gamma \in L : p_2(\gamma) \in W\}, L_{W'} = \{\gamma \in L : p_2(\gamma) \in W'\}$.

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- We show $L_W \overset{\text{BDE}}{\sim} L_{W'}$ using the following.

Lemma 7

Let $A, B \subset \mathbb{Z}^r$ and suppose that $A \times \mathbb{Z}^s \overset{BDE}{\sim} B \times \mathbb{Z}^s$ with constant K.

Then also $A \overset{BDE}{\sim} B$ with the same constant K.

This is proved using Hall's marriage theorem.

Translating the windows preserves $L_{W-x} \stackrel{\text{BDE}}{\sim} L_{W'-x}$

Lemma 8

$L_{W-x} \stackrel{BDE}{\sim} L_{W'-x}$ with the same constant K for every $x \in \mathbb{R}^n$ satisfying

$$(\partial W - x) \cap p_2(\Gamma) = (\partial W' - x) \cap p_2(\Gamma) = \varnothing. \tag{1}$$

- Since $|\partial W| = |\partial W'| = 0$ this is for a.e. $x \in \mathbb{R}^n$.
- Also proved using Hall's theorem.

Equidecomposition from the BDE pairing

- $F := \{\gamma \in L : |\gamma| \leqslant K\}$ with $L_{W-x} \overset{\text{BDE}}{\sim} L_{W'-x}$ with const. K.
- Equidecomposition defined on G cosets:

$$
\begin{array}{ccc}
L_{W-x} & \xrightarrow{\gamma \to p_2(\gamma)+x \bmod H} & A = W \cap (G+x) \\
\text{BDE pairing} \Big\downarrow F \text{ translates} & & \Big\downarrow p_2(F) \text{ equidecomposition} \\
L_{W'-x} & \xrightarrow[\gamma \to p_2(\gamma)+x \bmod H]{} & B = W' \cap (G+x)
\end{array}
$$

Equidecomposition from the BDE pairing

- $F := \{\gamma \in L : |\gamma| \leqslant K\}$ with $L_{W-x} \overset{\text{BDE}}{\sim} L_{W'-x}$ with const. K.
- Equidecomposition defined on G cosets:

$$\begin{array}{ccc} L_{W-x} & \xrightarrow{\gamma \to p_2(\gamma)+x \bmod H} & A = W \cap (G+x) \\ \text{BDE pairing} \downarrow F \text{ translates} & & \downarrow p_2(F) \text{ equidecomposition} \\ L_{W'-x} & \xrightarrow[\gamma \to p_2(\gamma)+x \bmod H]{} & B = W' \cap (G+x) \end{array}$$

- It remains to establish measurability of the pieces.

G-UNIFORMITY

- Decompose $G = \underbrace{e_1\mathbb{Z} \oplus \cdots \oplus e_d\mathbb{Z}}_{M,\text{ the free part}} \oplus \Delta$, with $|\Delta| < \infty$.
- Write $P_k = \left\{\sum_{j=1}^{d} m_j e_j : \ m_j \in \{0, 1, \ldots, k-1\}\right\}$ and $F_k = P_k \oplus \Delta$.
- Must show $\exists c > 0, k_0 > 0$ s.t. for almost all $x \in X$, all $k > k_0$

$$|W \cap (F_k + x)| \geqslant ck^d.$$

(Same for W'.)

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$$|W \cap (F_k + x)| \geqslant ck^d.$$

  (Same for W'.)
- $p_2(\Gamma)$ dense in $\mathbb{R}^n \implies G = p_2(\Gamma)/H$ dense in $X \implies M$ dense in X.
- $W \supseteq U =$ open ball of radius $\epsilon > 0$. P_{k_0} is an $\epsilon/2$-net in X for some k_0.
- $P_k + x$ contains at least $\lfloor k/k_0 \rfloor^d$ translated copies of $P_{k_0} + x$.
- $P_k + x$ contains at least ck^d points in W.

Without cut-and-project sets

- Recently my PhD student Manos Spyridakis has given a proof without going through cut-and-project sets.
- Again the Measurable Hall's theorem is used.

The end

Thank you!